Phase transitions in a simple growth model for a driven interface in random media

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We introduce a simple growth model for a driven interface in random media, exhibiting a smoothing (roughening) transition as well as a pinning-depinning transition in a nonequilibrium $(1+1)$-dimensional system. At both transition points, the scaling exponents belong to the directed percolation universality class. The rough interface at the pinning-depinning transition point belongs to the quenched Kardar-Parisi-Zhang universality class. The two transitions are second order phase transitions. We also introduce a modified growth model exhibiting the pinning-depinning transition. In the modified model, the pinning-depinning transition is a first order phase transition in the directed percolation universality class.

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The problem of phase transitions in nonequilibrium $(1+1)$-dimensional systems has recently attracted much interest [1–3]. Usually, phase transitions in nonequilibrium $(1+1)$-dimensional systems are observed in systems with absorbing states. These systems exhibit a phase transition from an active to an inactive (absorbing) state. Examples are the monomer-dimer model for the catalytic oxidation of CO [4], the contact process [5], branch-annihilation random walks with odd numbers of offspring [6], the interacting monomer-dimer model [2], etc. It is well known that this phase transition is related to directed percolation (DP) [7] or the parity conserving (PC) universality class [2,6,8–10]. DP is the generic universality class for phase transitions from active to inactive states, and PC is related to the phase transitions in a few models with two symmetric absorbing states. The representative example for the PC universality class is branching-annihilating random walkers with an even number of offspring [6].

Recently, it has been reported that DP or PC in a few models is related to the roughening transition of a growing interface in a nonequilibrium $(1+1)$-dimensional system. An interface under thermal equilibrium in $1+1$ dimensions is always rough and thus does not exhibit a roughening transition from a smooth phase to a rough one with diverging width or vice versa. In higher dimensions an interface under thermal equilibrium can undergo a roughening transition at some critical temperature. However, a surface far from equilibrium in $1+1$ dimensions can exhibit a nonequilibrium roughening (NR) transition, although there are few examples [1,11,12]. These examples are polynuclear growth models [12], the fungal growth model [13], the solid-on-solid model with evaporation at the edge of the terrace [1], and the dimer adsorption-desorption model [3]. The important feature of the NR transition is its relation to the DP or PC class at a specific reference height of the interface. All NR transitions in a $(1+1)$-dimensional system have been observed in interfaces fluctuating in homogeneous media. Thus it would be interesting to find a model describing a driven interface in random media that exhibits a nonequilibrium roughening transition.

An important feature of the motion of a driven interface in random media is the interplay between the quenched disorder and the driving force acting on the interface. The interface is pinned when the driving force F is smaller than the pinning strength induced by the quenched disorder. For a large F, however, the interface can move for a while until it is pinned again. There exists a threshold of the driving force F_p above which the interface moves with a finite velocity. Accordingly the velocity is zero for $F<F_p$ and it is nonzero for $F>F_p$. This phenomenon is called *a pinning-depinning transition*. When $F>F_p$, we expect $v\sim(F-F_p)^{\theta}$, where θ is the velocity exponent.

The dynamics of driven interfaces in a random medium has been well explained by the quenched Kardar-Parisi-Zhang (QKPZ) equation [14],

$$\frac{\partial h(x,t)}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+F+\eta(x,h), \tag{1}$$

where $h(x,t)$ is the height of the interface at position x and time t. F is an external driving force and η is a quenched noise with $\langle \eta(x,h)\rangle=0$ and $\langle \eta(x,h)\eta(x',h')\rangle=2D\delta(x-x')\delta(h-h')$. In the QKPZ equation, the pinning-depinning transition is a second order phase transition and the value of the velocity exponent is about 0.636 [14]. The fluctuating interface formed by the QKPZ equation is always rough in the regime $F>F_p$ so that no NR transition has been observed yet.

In this paper we introduce a simple growth model exhibiting not only a pinning-depinning transition but also a smoothing (roughening) transition in nonequilibrium $(1+1)$-dimensional systems. In our model the interface is driven in a random medium by a driving force F. The scaling properties of the model at both transition points are related to the DP class. We find that the pinning-depinning transition is a second order phase transition and the scaling exponent near the threshold F_p belongs to the QKPZ universality class. In addition to the depinning transition a smoothing transition occurs at F_s for $F_s>F_p$. The interface maintains a smooth phase for $F>F_s$. We find that the smoothing transition is also a second order phase transition.

The model is defined on a $(1+1)$-dimensional lattice with a periodic boundary condition. We consider a two dimensional checkerboard lattice, rotated at 45° to the square lattice. Every site in the lattice can be occupied or vacant. We define the interface as a borderline between the occupied

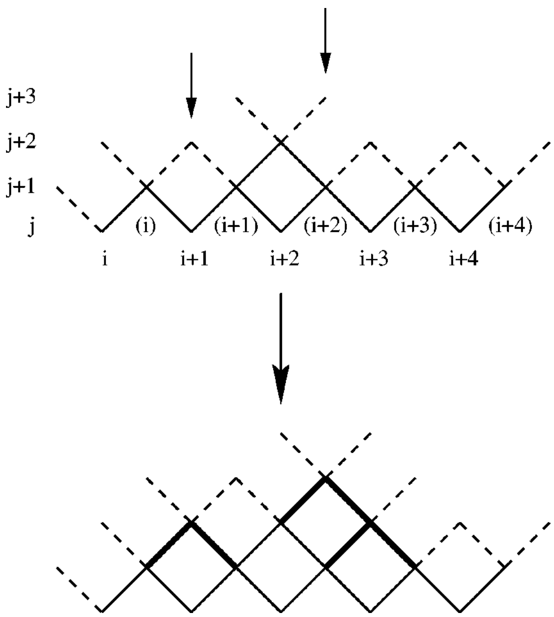

FIG. 1. Schematic representation of the stochastic growth rule of the model. The two small arrows indicate the sites having smaller random numbers than the driving force. The bold lines denote the newly updated lines on the interface. The i's and (i)'s denote sites on the substrate at the height values j and $j+1$, respectively. Top: The height configuration $H(t)$ before growth at time t is $H(t) = \{\ldots, h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}, \ldots\} = \{\ldots, j, j, j+1, j, j, \ldots\}$. Bottom: The $H(t+1)$ after growth is $H(t+1) = \{\ldots, h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}, \ldots\} = \{\ldots, j, j+1, j+2, j+1, j, \ldots\}$.

and vacant sites. We preassign random numbers between 0 and 1, representing impurities in a random medium, to all perimeter sites of the initially flat interface. A constant driving force F is thus applied to the interface. Each site on the interface has one or two nearest neighbor vacant sites in the direction of the driving force, which can become occupied at each time step. A vacant site i is occupied when the value of the random number at the site is smaller than the driving force F. At each time, the growth of the interface is made by parallel updating of all the nearest neighboring vacant sites of the interface. After the growth, we impose the restricted solid-on-solid (RSOS) condition $|h_i - h_{i+1}| \leqslant 1$ on all sites on the interface. Here, h_i means the height at the site i. The RSOS condition is fulfilled by instantaneous avalanche processes after parallel updating (see Fig. 1).

Although the growth rule of our model is simple, the model exhibits rich critical behaviors. For small values of the driving force F, the interface is pinned temporarily after some movements. The interface shows the same behavior for $F < F_p$. The velocity of the interface, however, becomes finite for $F > F_p$. We carried out the simulation for the system size $L = 10\,000$. The velocity versus the driving force is plotted in Fig. 2. By fitting the velocity data above the threshold to $v \sim (F - F_p)^\theta$, we obtained the critical driving force as $F_p \simeq 0.34$ and $\theta = 0.623 \pm 0.004$ in the QKPZ universality class [14].

Near the depinning transition, the dynamics shows nontrivial scaling behavior in the global interface width. The global interface width, defined by $W(L,t) = \langle L^{-d'} \Sigma_i [h_i(t) - \overline{h}(t)]^2 \rangle^{1/2}$, scales as

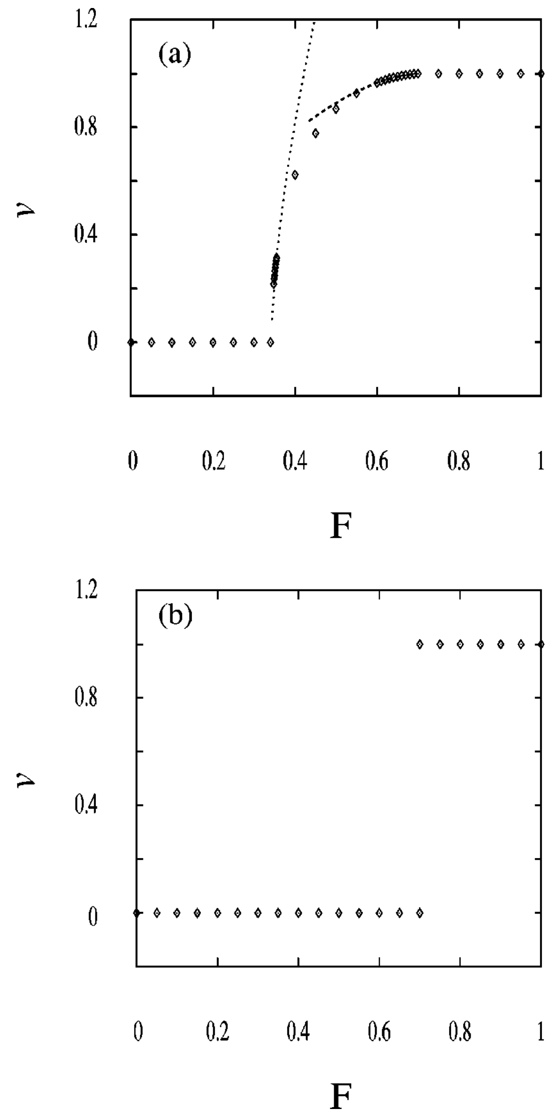

FIG. 2. Plot of average velocity versus external driving force for a system size $L = 10\,000$ [for the orginal model (a) and the modified model (b)]. The dotted line (bottom) in (a) is $v \sim (F - 0.34)^{0.623}$. The dotted line (top) in (a) is $V(=1-v) = 1 - 1.75(0.7055 - F)^{1.772}$.

$$W(L,t) \sim \begin{cases} t^{\zeta/z} & \text{if} \quad t \ll L^z \\ L^\zeta & \text{if} \quad t \gg L^z. \end{cases} \tag{2}$$

Here $\overline{h}$, L, d', and $h_i(t)$ denote the mean height, system size, substrate dimension, and the height at time t and site i, respectively. ζ and z are called the roughness and the dynamic exponents. The roughness exponent ζ can also be obtained from the height-height correlation function $C(x) = \langle (h_{i+x} - h_i)^2 \rangle^{1/2} \sim x^\zeta$ in the long time limit. Near the depinning threshold $F_p = 0.34$, the roughness exponent is measured to be $\zeta = 0.630 \pm 0.001$ from the height-height correlation function. The growth exponent ζ/z from the global width is obtained as 0.72 ± 0.01. The roughness exponent obtained is very close to the value for the QKPZ universality class, $\zeta = 0.633 \pm 0.001$ [14]. A notable growth model mimicking the QKPZ equation near the threshold F_p was proposed by Sneppen several years ago [15]. From the model, it has naturally been concluded that the interface at the threshold of the depinning transition can be described by a DP cluster spanned perpendicular to the interface growth direction in $1+1$ dimensions. The roughness exponent ζ of the interface is given by the ratio of the correlation length exponents $\nu_\perp$ and $\nu_\parallel$ of the DP cluster in the transverse and longitudinal directions, that is, $\zeta=\nu_\perp/\nu_\parallel=0.633\pm0.001$. The dynamic rule of the Sneppen model is the same as in our model at the depinning threshold. Accordingly, the dynamics of the model belongs to the DP universality class.

In the depinned phase ($F>F_p$), the interface grows with a finite velocity, which increases as the driving force F does until $F_s=0.7055$. Surprisingly, at the critical driving force F_s, the velocity of the driven interface saturates to 1. Although the number of newly updated sites per each Monte Carlo time step is smaller than the system size L at F_s, the velocity of the interface can become 1 because of the avalanche process satisfying the RSOS condition. At the same time, the roughness of the interface decreases as the driving force F increases and the value of the roughness exponent becomes 0 at F_s, indicating a smooth phase. This nonequilibrium smoothing (roughening) transition at F_s is also related to the DP class. It has been reported that, in models exhibiting a NR transition in $1+1$ dimensions, the DP process emerges at a particular reference height of the interface [13]. In those models, the reference height is the bottom layer of the interface. The sites where the interface touches the reference height correspond to the active sites of DP. Therefore, in the active phase of DP the interface fluctuates close to the reference level so that the interface is smooth. On the other hand, in the inactive phase of DP, the interface detaches from the reference level and evolves into a rough state. In our model, the sites whose height is the same as the Monte Carlo time correspond to the active sites of DP. The level of the reference height in our model is the Monte Carlo time at each time step and so always varies as time elapses.

We examined the scaling behavior of the interface width at the critical driving force F_s. The width at F_s grows as $W_c\sim(\ln t)^{\gamma_1}$ before saturation. After saturation, the width is $W_c\sim(\ln L)^{\gamma_2}$. Here γ_1 and γ_2 are obtained as $\gamma_1\simeq0.33$ and $\gamma_2\simeq0.48$. In the polynuclear growth model introduced by Kertész and Wolf [12], which shows a NR transition in $1+1$ dimensions, the values of γ are $\gamma_1=\gamma_2\simeq0.5$. However, in the restricted and unrestricted models by Alon *et al.* [1], the values of γ are $\gamma_1=\gamma_2\simeq0.43$ and $\gamma_1=\gamma_2\simeq0.25$, respectively. These facts indicate that the width at F_s shows nonuniversal behavior. The fact that $W_c\sim(\ln L)^{\gamma_2}$ indicates that the morphology of the moving interface for $F>F_s$ is smooth. Usually, in a model exhibiting the NR transition in $1+1$ dimensions, the smooth phase corresponds to the active DP phase, whereas the rough phase corresponds to the nonactive DP phase. In our model the occupied sites on the interface, whose height value is the Monte Carlo time, can be considered as the active sites of a DP process. We measured the density of occupied sites, $\rho(F,t)$, at the reference height. $\rho(F,t)$ is saturated at a finite value for $F>F_s$ and decreases to zero exponentially for $F<F_s$ in the long time limit. The result is shown in Fig. 3. At F_s, $\rho(F_s,t)$ scales as

$$\rho(F_s,t)\sim t^{-\beta/\nu_\parallel}. \tag{3}$$

From the Monte Carlo simulation for different system sizes $L=64$–1024, we measured $\beta/\nu_\parallel=0.1596\pm0.002$, which is in excellent agreement with the DP value 0.1595 [16]. We also considered the density $\rho_s(F_s,t)$, which is averaged over samples with at least one occupied site at the reference height. The density decays as Eq. (3) before the saturation time $\tau(t<\tau)$ and has a finite value for $t>\tau$. The steady state value of $\rho_s(F_s)$ depends on system size L as $\rho_s(F_s)\sim L^{-\beta/\nu_\perp}$. We obtained $\beta/\nu_\perp=0.253\pm0.002$, which is almost the same as the expected value from the DP class (see Fig. 4), 0.252 [16].

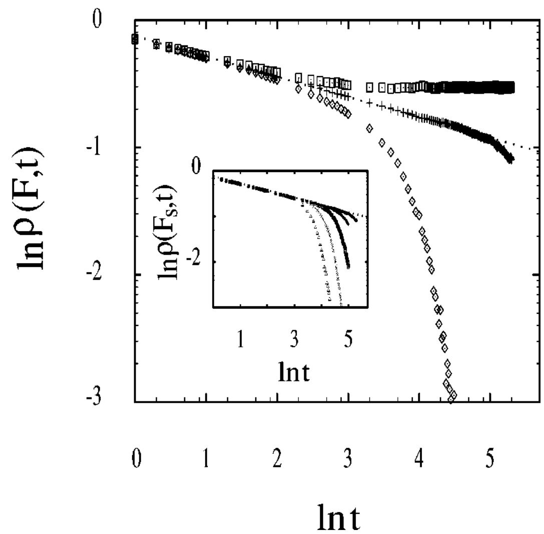

FIG. 3. Plot of $\rho(F,t)$ versus t in double logarithmic scale for the driving force $F=0.707$ (top), $0.7055(=F_s)$, and 0.702 (bottom). The data were obtained for a system size 1024. The line obtained from the least squares fit has the slope $\beta/\nu_\parallel=0.1596$. Inset: Plot of $\rho(F_s,t)$ versus t in double logarithmic scale for the system sizes $L=64$, 128, 256, 512, and 1024 at the critical driving force F_s.

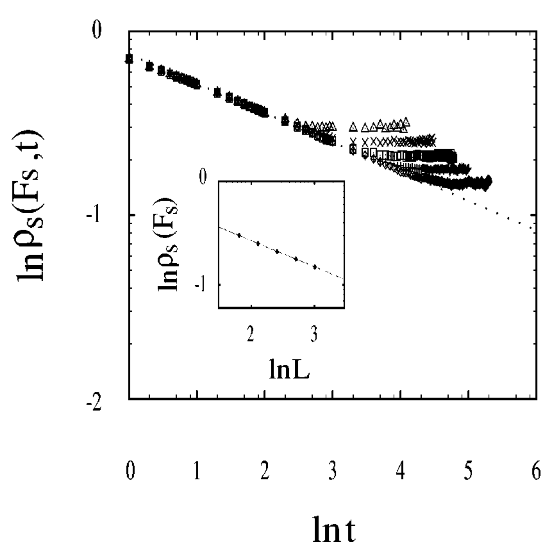

FIG. 4. Plot of $\rho_s(F_s,t)$ versus t in double logarithmic scale for the system sizes $L=64$, 128, 256, 512, and 1024. The slope of the dotted line is $\beta/\nu_\parallel=0.1596$. Inset: Plot of $\rho_s(F_s)$ versus L in double logarithmic scale for the system sizes $L=64$, 128, 256,512, and 1024. The line obtained from the least squares fit has the slope $\beta/\nu_\perp=0.253$.

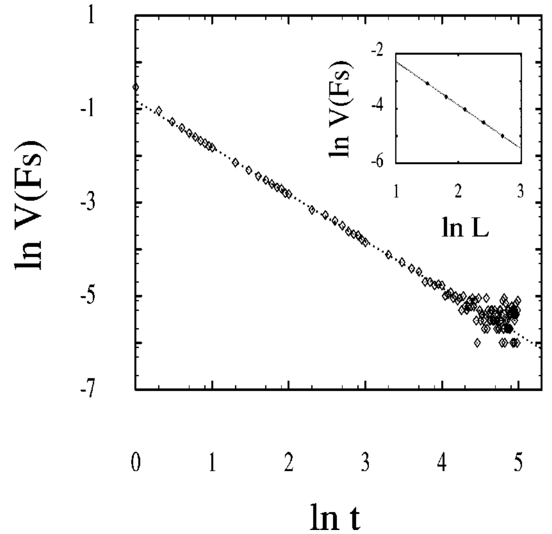

FIG. 5. Plot of V versus t in double logarithmic scale for the system size 1024. The slope of the dotted line is $\alpha=1.0$. Inset: Plot of $V(F_s)$ versus L in double logarithmic scale for the system size $L=32$, 64, 128, 256, and 512. The line obtained from the least squares fit has the slope $\nu_\parallel/\nu_\perp=1.592$.

Next, we define a convenient order parameter for the smoothing transition as $V=1-v$, where v denotes the mean velocity of the interface. The order parameter V is zero in the smooth phase and is not zero in the rough phase (see Fig. 2). The order parameter is characterized by the inverse of the characteristic time T, $V\sim 1/T$, where T is regarded as the characteristic time that the DP correlations survive. The characteristic time T is in proportion to the correlation length $\xi_\parallel$ of the DP cluster in the longitudinal direction. Hence, $V\sim 1/T\sim \epsilon^{\nu_\parallel}\sim \xi_\perp^{-\nu_\parallel/\nu_\perp}$, where $\epsilon=F_s-F\ll 1$ and $\xi_\perp$ denotes the correlation length of the DP cluster in the transverse direction. The order parameter V is shown in Fig. 5. By fitting the velocity data below the threshold F_s to $V\sim\epsilon^{\nu_\parallel}$, we obtained $\nu_\parallel=1.772\pm 0.006$, in good agreement with the DP universality class, 1.736 [16]. The order parameter V decays as $V\sim 1/t^{\alpha}(\alpha=1)$ for time $t<T$ and has a finite value for $t>T$. The steady state value of V at F_s depends on system size L as $V(F_s)\sim L^{-\nu_\parallel/\nu_\perp}$. At F_s we measured $V\sim t^{-1.00\pm 0.04}$ for $t<T$ and $V\sim L^{-1.592\pm 0.007}$ for $t>T$, in good agreement with the DP theory (see Fig. 5), 1.582 [16].

Recently Alon *et al.* [1]. showed, through the study of a stochastic growth model exhibiting the NR transition, that spontaneous symmetry breaking (SSB) may take place in nonequilibrium situations under certain conditions in 1+1 dimensions. It would be interesting to consider whether our model shows the same SSB as that found by Alon *et al.* They defined a magnetizationlike order parameter to quantify SSB in their model, as follows:

$$M=\frac{1}{L}\sum_{i=1}^{L}(-1)^{h_i}. \tag{4}$$

They found $\langle M\rangle\neq 0$ in the smooth phase and $\langle M\rangle=0$ in the rough phase. In their model, the interface in the smooth phase becomes pinned at a specific height in the long time limit and the velocity becomes zero in the thermodynamic limit. Only in a finite system will the interface have a finite velocity that vanishes exponentially with the system size. This breaks the symmetry between odd and even heights in the thermodynamic limit. The final interface will be pinned at an odd or even height, depending on the initial conditions, breaking the symmetry of the dynamics. On the other hand, in our model the velocity of the interface is always finite in the smooth phase, not allowing SSB. Therefore, there is no SSB in our model like that found by Alon *et al.* [1].

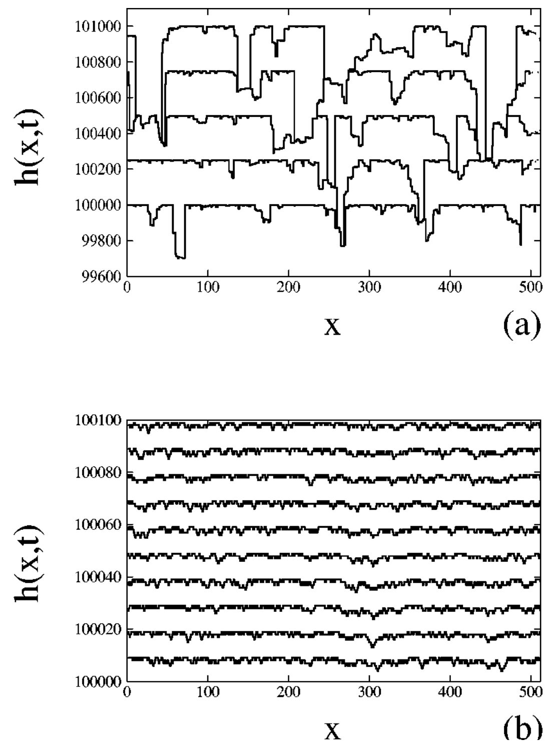

FIG. 6. Snapshots of the interfaces formed from the modified model (a) and the orginal model (b) for the driving force 0.7055 $(=F_s)$. The data are obtained for a system size 512. The growing interface is rough in the modified model at F_s, but smooth in the original model.

We then considered a modified model that has the same growth rule as that of the original model, but without the avalanche process satisfying the RSOS condition. The growth in the modified model occurs only at the sites whose height is the same as the reference height, i.e., the Monte Carlo time. We found that the greatest height of the interface is lower than the reference height for $F<F_s$, but there exists at least one site whose height is the same as the reference height at F_s. Therefore, one can deduce that the critical depinning force is also $F_s=0.7055$ in the modified model. When $F<F_s$, the interface in our modified model is pinned and the velocity is thus 0. An interesting feature is that the velocity of the interface jumps from 0 to 1 at $F=F_s$. In the modified model, there is no avalanche process which makes the velocity of the growing interface 1 at F_s, but the height increment of the interface at newly updated sites can be bigger than one lattice unit. In the original model, the height increment at the newly updated site is always one lattice unit. This large height increment plays the same role as the avalanche process in the original model, and so the velocity of the growing interface in the modified model becomes 1 at F_s. The pinning-depinning transition in the modified model is, therefore, a *first order* phase transition. The velocity ver-

sus the driving force is plotted in Fig. 2(b). $\rho(F_s,t)$ and $\rho_s(F_s)$ in the modified model show the same DP behavior as that of the original model. It is a very interesting fact that a first order phase transition is caused by DP behavior.

Unlike the modified model, it is possible in the original model for growth to occur at sites on the interface that have lower heights than the reference one. The reason is because avalanche processes satisfying the RSOS condition change the distribution of random numbers of already pinned sites in the interface. The avalanche process makes an interface with lower height than the reference height continue growing. The growth induced by the avalanche process thus decreases the velocity of the growing interface gradually from 1 as the driving force decreases from $F=F_s$. At $F=F_p$ the interface is pinned. Avalanche processes, therefore, prevent the velocity of the interface from dropping suddenly to zero at F_s, and make the velocity decrease continuously to zero. This fact makes both the smoothing and pinning-depinning transitions become second order phase transitions in the original model.

At F_s, both the original and modified models show the same DP behavior for $\rho(F_s,t)$ and $\rho_s(F_s)$, but the morphologies of interfaces formed from the two models are quite different from each other. The morphology of the interface in the original model is smooth, but the morphology in the modified model is quite rough at F_s. A snapshot of both interfaces is compared in Fig. 6.

In conclusion, we have introduced a simple growth model for a driven interface in random media, exhibiting a smoothing (roughening) transition as well as a pinning-depinning transition in a nonequilibrium $(1+1)$-dimensional system. The model shows the pinning-depinning transition at $F_p=0.34$. The transition is a second order phase transition. We found that the dynamics of the moving interface near the depinning threshold belongs to the quenched Kardar-Parisi-Zhang universality class. Our model then shows a smoothing (roughening) transition at $F_s=0.7055$. The roughness exponent of the moving interface formed by the model is 0 at F_s. We found that, at the transition point, the scaling exponents characterizing the smoothing transition belong to the directed percolation universality class. We also introduced a modified model without the RSOS condition in the process of interface growth. We found that the modified model shows the pinning-depinning transition at F_s rather than F_p. The pinning-depinning transition is a first order phase transition.

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[1] U. Alon, M.R. Evans, H. Hinrichsen, and D. Mukamel, Phys. Rev. Lett. **76**, 2746 (1996); Phys. Rev. E **57**, 4997 (1998).
[2] M.H. Kim and H. Park, Phys. Rev. Lett. **73**, 2579 (1994); H. Park, M.H. Kim, and H. Park, Phys. Rev. E **52**, 5664 (1995).
[3] H. Hinrichsen and G. Ódor, Phys. Rev. Lett. **82**, 1205 (1999).
[4] R.M. Ziff, E. Gulari, and Y. Barshad, Phys. Rev. Lett. **56**, 2553 (1985).
[5] T.E. Harris, Ann. Prob. **2**, 969 (1974); T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[6] H. Takayasu and A. Yu Tretyakov, Phys. Rev. Lett. **68**, 3060 (1992).
[7] G. Deutscher, R. Zallen, and J. Adler, *Percolation Structures and Processes*, Annals of the Israel Physical Society, Vol. 5 (Adam Hilger, Bristol, 1983).
[8] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A **17**, L105 (1984); P. Grassberger, *ibid.* **22**, L1103 (1989).
[9] N. Menyhárd, J. Phys. A **27**, 6139 (1994); N. Menyhárd and G. Ódor, *ibid.* **29**, 7739 (1996).
[10] K.E. Bassler and D.A. Browne, Phys. Rev. Lett. **77**, 4094 (1996); Phys. Rev. E **55**, 5225 (1997); K.S. Brown, K.E. Bassler, and D.A. Browne, *ibid.* **56**, 3953 (1997).
[11] R. Savit and R. Ziff, Phys. Rev. Lett. **55**, 2515 (1985).
[12] J. Kertész and D.E. Wolf, Phys. Rev. Lett. **62**, 2571 (1989).
[13] J.M. Lopéz and H.J. Jensen, Phys. Rev. Lett. **81**, 1734 (1998).
[14] A.-L. Barabási and H.E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).
[15] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).
[16] V. Privman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University, Cambridge, England, 1996).